Optimal Multilevel Solvers and New Hybridized Mixed Methods for Linear Elasticity

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Abstract

We present a family of new mixed finite element methods for the linear elasticity and then develop multilevel solvers to solve the linear system. Our mixed discretization, preserving the symmetry and the $H(\mathrm{div})$ conformity in the stress approximation, can be efficiently implemented by hybridization, which reduces the indefinite system to a symmetric positive-semidefinite system. The condition number of the reduced system, which is characterized by a non-inherited bilinear form, depends not only on the grid size but also on the material parameters and the singularity of the grids. By constructing uniformly stable interpolation operators between the non-nested spaces, we prove that our multilevel solvers converge uniformly with respect to both the grid size and Poisson's ratio. Numerical experiments are presented to validate our theoretical results.

Motivation

Some biological soft tissues, such as the arterial wall, are nearly incompressible and are reinforced by **collagen fibers**, which induce the anisotropy in the mechanical response. To obtain physiologically realistic models for such materials, one need to understand the orientations of the fibers. However, the in-vivo identification of patient-specific fiber orientations in real arteries, whose geometry significantly differs from an idealized thick-walled tube, in particular for atherosclerotic arteries, is difficult. Another promising scheme for the automated calculation of the fiber orientations is based on the assumption that the fiber orientations are mainly governed by the principal tensile stress directions resulting in an improved load transfer within the artery. This scheme requires the numerical approximation for the symmetric stress and then the computation of the principal stress.

The mixed finite element methods are popular in solid mechanics since they avoid locking and provide a straightforward approximation for stress. However, the construction of stable finite element spaces for the classical Hellinger-Reissner variational formulation using polynomial shape functions is very challenging. In the following, we consider the following variational problem: Find $(\boldsymbol{\sigma}, u) \in \Sigma \times V := H(\mathrm{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^n)$ such that

$$\begin{cases} (\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \Sigma, \\ (\mathrm{div}\boldsymbol{\sigma}, v) = (f, v) & \forall v \in V. \end{cases} \tag{1}$$

Main Objectives and Difficulties

- Objective 1: High-precision and structure-preserving approaches for stress analysis.
- Objective 2: Robust and scalable iterative solvers for nearly incompressible materials.
- Difficulty 1: large system ☹
  - High order ($k \geq n$) conforming elements: Hu-Zhang (2014, 2015)
  - Lowest conforming elements in 3D: $\mathcal{P}_4(\mathbb{S}) - \mathcal{P}_3(\mathbb{R}^3)$, number of local d.o.f.

$$6C_7^3 + 3C_6^3 = 210 + 60 = 270.$$

- Difficulty 2: hard to design solver for mixed formulation ☹
- Difficulty 3: nearly incompressible material ☹

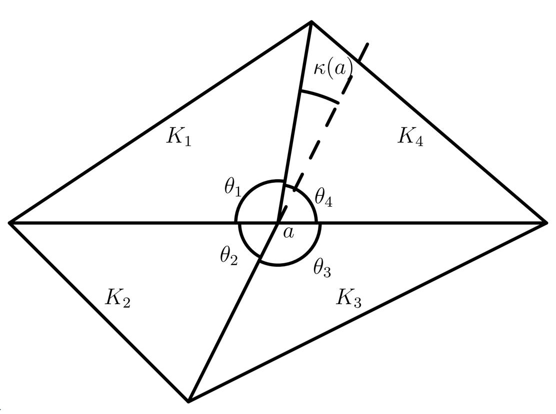


Figure 1: A nearly singular vertex

Mixed Methods and Hybridization

- Discrete stress space and discrete displacement space

$$\Sigma_{h,k+1} = \{\boldsymbol{\tau} \in H(\mathrm{div}, \mathbb{S}) \mid \boldsymbol{\tau}|_K \in P_{k+1}(K; \mathbb{S}) \quad \forall K \in \mathcal{T}_h\}.$$

$$V_{h,k} = \{v \in L^2(\mathbb{R}^d) \mid v|_K \in P_k(K, \mathbb{R}^d) \quad \forall K \in \mathcal{T}_h\}.$$

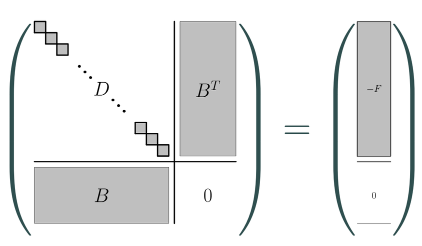
- Hybridization: discontinuous stress space + lagrange multiplier space

$$\Sigma_{h,k+1}^{-1} = \{\boldsymbol{\tau}_h \in L^2(\mathbb{S}) \mid \boldsymbol{\tau}_h|_K \in P_{k+1}(K; \mathbb{S}) \quad \forall K \in \mathcal{T}_h\}.$$

$$M_{h,k+1} = \{\mu_h \in L^2(\mathcal{F}_h; \mathbb{R}^d) \mid \mu_h|_F = P_{k+1}(F, \mathbb{R}^d) \quad \forall F \in \mathcal{F}_h^i \text{ and } \mu|_{\mathcal{F}_h^\partial} = 0\}$$

- Hybridized mixed method: find $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \Sigma_{h,k+1}^{-1} \times V_{h,k} \times M_{h,k+1}$ such that

$$\begin{aligned} (\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\mathrm{div}\boldsymbol{\tau}_h, u_h) + \langle [\boldsymbol{\tau}_h], \lambda_h \rangle_{\mathcal{F}_h^i} &= 0 & \forall \boldsymbol{\tau}_h \in \Sigma_{h,k+1}^{-1}, \\ (\mathrm{div}\boldsymbol{\sigma}_h, v_h) &= (f, v_h) & \forall v_h \in V_{h,k}, \\ \langle [\boldsymbol{\sigma}_h], \mu_h \rangle_{\mathcal{F}_h^i} &= 0 & \forall \mu_h \in M_{h,k+1}. \end{aligned}$$



Theorem 0.1. *Let $(\boldsymbol{\sigma}, u) \in \Sigma \times V$ be the exact solution of the problem (1) and $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_{h,k+1} \times V_{h,k}$ the finite element solution. Assume that $\boldsymbol{\sigma} \in H^{k+2}(\Omega; \mathbb{S})$ and $u \in H^{k+1}(\Omega; \mathbb{R}^n)$. We have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathrm{div})} + \|u - u_h\|_0 \lesssim h^{k+1}(|\boldsymbol{\sigma}|_{k+2} + |u|_{k+1}). \tag{2}$$

Theorem 0.2. *The Lagrange multiplier λ_h satisfies*

$$s(\lambda_h, \mu_h) = -(f, u_{\mu_h}) \qquad \forall \mu \in M_{h,k+1}, \tag{3}$$

where $s(\lambda_h, \mu_h) = (\mathcal{A}\boldsymbol{\sigma}_{\lambda_h}, \boldsymbol{\sigma}_{\mu_h})$. Moreover, the system (3) is symmetric positive-semidefinite and its kernel is $\mathrm{R}(\mathcal{C})^\perp$. We also have the condition number estimate:

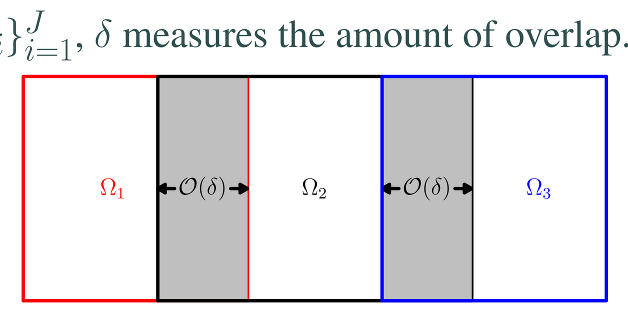
$$\mathrm{cond}(S) \lesssim \frac{2\tilde{\mu} + \tilde{\lambda}}{2\tilde{\mu}} h^{-2} \sin^{-2}(\kappa_0). \tag{4}$$

Multilevel Solvers

- Choosing a good smoother, appropriate coarse-scale problems and inter-scale transfer operators.
- Constructing coarse-scale approximations to the fine-scale variables.

Schwarz Smoother

- Overlapping subdomains $\{\Omega_i\}_{i=1}^J$, δ measures the amount of overlap.



- Subspaces for $1 \leq i \leq J$, $M_i = \{\lambda \in M_{h,k+1} \mid \lambda|_F = 0 \ \ \forall F \in \mathcal{F}_h \backslash \Omega_i^0\}$.
- Subproblems $S_i : M_i \mapsto M_i'$, where $\langle S_i \lambda_i, \mu_i \rangle := s(\iota_i \lambda_i, \iota_i \mu_i)$

Coarse Problem

- Norm equivalence

$$\|\lambda\|_S^2 \eqsim 2\tilde{\mu}|\lambda|_h^2 + \tilde{\lambda}|\lambda|_*^2 \qquad \forall \lambda \in M_{h,k+1}.$$

- Comparing to the primal elasticity

$$\|w_H\|_{A_H}^2 \eqsim 2\tilde{\mu}|\epsilon(w_H)|_0^2 + \tilde{\lambda}\|P_0^H \mathrm{div} w_H\|_0^2,$$

with $\langle A_H w_H, v_H \rangle = 2\tilde{\mu}(\epsilon(w_H), \epsilon(v_H)) + \tilde{\lambda}(P_0^H \mathrm{div} w_H, P_0^H \mathrm{div} v_H)$,

- Key idea: Using the Lagrange element $\mathcal{P}_2$ as the coarse space.

$$W_H := \{w \in H_0^1(\Omega; \mathbb{R}^2) \mid w|_K \in \mathcal{P}_2(K; \mathbb{R}^2) \text{ for } K \in \mathcal{T}_H\}.$$

Intergrid Transfer operator

1. Step 1: Harmonic extension: the harmonic extension $\tilde{I}_H^h : W_H \mapsto W_h$ (Schöberl, 1999). On each edge of coarse element $K_H \in \mathcal{T}_H$

$$\tilde{I}_H^h w_H|_{\partial K_H} = w_H|_{\partial K_H}, \qquad a_h(\tilde{I}_H^h w_H, v_h) = 0 \quad \forall v_h \in W_{h,0}(K_H). \tag{5}$$

2. Step 2: $Q_h : W_h \mapsto M_{h,k+1}$, L^2 projection.
3. The intergrid transfer operator I_H^h:

$$I_H^h := Q_h \tilde{I}_H^h : W_H \mapsto M_{h,k+1}. \tag{6}$$

Main Theorem

A multilevel preconditioner is $B = I_H^h \tilde{A}_H^{-1}(I_H^h)' + \sum_{i=1}^J \iota_i S_i^{-1} \iota_i'$, where $\tilde{A}_H$ is a multilevel approximation for A_H.

Theorem 0.3. *The condition number of BS satisfies*

$$\mathrm{cond}(BS) \leq C(1 + N_c)\frac{H^2}{\delta^2},$$

where C is independent to both the mesh size h and the Lamé constants.

Numerical Results

- Convergence tests for the lowest order:

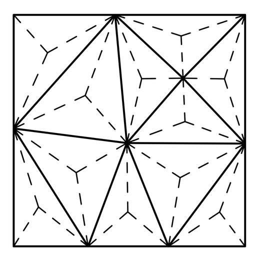


Figure 2: A macro-simplex grid

$1/h$	$\|u - u_h\|_0$	Order	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0$	Order	$\|\mathrm{div}\boldsymbol{\sigma} - \mathrm{div}\boldsymbol{\sigma}_h\|_0$	Order
4	9.5309e-2	–	2.0147e-1	–	2.6589e-0	–
8	4.5289e-2	1.07	4.9971e-2	2.01	1.2995e-0	1.03
16	2.2009e-2	1.04	1.2357e-2	2.01	6.3735e-1	1.02
32	1.0976e-2	1.00	3.1761e-3	1.96	3.1827e-1	1.00
64	5.4797e-3	1.00	8.0961e-4	1.97	1.5892e-1	1.00

Table 1: Errors and observed convergence orders on macro-simplex grids, $k = 0$

- Convergence tests for the high order:

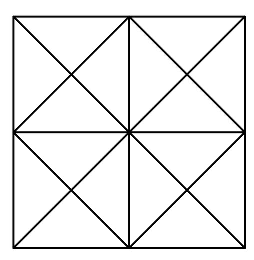


Figure 3: A crisscross grid

$1/h$	$\|u - u_h\|_0$	Order	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0$	Order	$\|\mathrm{div}\boldsymbol{\sigma} - \mathrm{div}\boldsymbol{\sigma}_h\|_0$	Order
4	5.7633e-4	–	3.1371e-4	–	1.7027e-2	–
8	7.2355e-5	2.99	2.0057e-5	3.96	2.1361e-3	2.99
16	9.0541e-6	2.99	1.2672e-6	3.98	2.6726e-4	2.99
32	1.1320e-6	3.00	7.9629e-8	3.99	3.3416e-4	3.00
64	1.4151e-7	3.00	4.9899e-9	4.00	4.1772e-5	3.00

Table 2: Errors and observed convergence orders on crisscross grids, $k = 2$

- Tests for two-level solvers and multilevel solver:

$1/h$ \ $\tilde{\nu}$	0.49	0.499	0.4999	0.49999	0.499999	0.4999999
4	17, 3	18, 4	21, 4	23, 4	23, 4	23, 4
8	17, 4	20, 4	25, 4	27, 5	28, 5	29, 5
16	18, 4	20, 4	26, 5	28, 5	29, 5	29, 5
32	18, 4	20, 4	25, 5	27, 5	28, 5	29, 5

Table 3: Number of iterations of PCG: Two-level additive (left) and multiplicative (right) preconditioner

$1/h$ \ $\tilde{\nu}$	0.49	0.499	0.4999	0.49999	0.499999	0.4999999
4	4	5	5	5	5	5
8	4	6	7	7	7	7
16	5	6	7	7	7	7
32	5	6	7	7	7	7

Table 4: Number of iterations of PCG, multilevel symmetrized multiplicative preconditioner

Conclusions

1. A family of mixed finite elements for elasticity,
2. Using hybridization to reduce the dimension of linear system,
3. The solution cost is dominated by solving a SPSD system,
4. Two-level and multilevel precoditioner using the primal formulation as the coarse problem.
5. Future works: singular vertex.

References

[1] **S. Gong**, S. Wu, and J. Xu. New Hybridized Mixed Methods for Linear Elasticity and Optimal Multilevel Solvers. submitted to *Numer. Math.*.

[2] **S. Gong**, X.-C. Cai. A nonlinear elimination preconditioned Newton's method with applications in arterial wall simulation. submitted to *Proceedings of the 24th International Conference on Domain Decomposition Methods*.

[3] S. Wu, **S. Gong**, and J. Xu. Interior penalty mixed finite element methods of any order in any dimension for linear elasticity with strongly symmetric stress tensor. accepted by *Math. Models Methods in Appl. Sci.*.

[4] W. Hao, **S. Gong**, S. Wu, J. Xu, M. R. Go, A. Friedman, and D. Zhu. A mathematical model of aortic aneurysm formation. *PloS one 12*, No. 2 (2017): e0170807.

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